Wegner bounds for a two-particle tight binding model

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Abstract

We consider a quantum two-particle system on a lattice $\mathbb{Z}^d$ with interaction and in presence of an IID external potential. We establish Wegner-typer estimates for such a model. The main tool used is Stollmann's lemma.

1 Introduction. The results

This paper considers a two-particle Anderson tight binding model on lattice $\mathbb{Z}^d$ with interaction. The Hamiltonian $H \left(= H_{U,V,g}^{(2)}(\omega) \right)$ is a lattice Schrödinger operator (LSO) of the form $H^0 + U + g(V_1 + V_2)$ acting on functions $\phi \in \ell_2(\mathbb{Z}^d \times \mathbb{Z}^d)$:

$$
\begin{aligned}
H\phi(\underline{x}) \; &= H^0\phi(\underline{x}) + \left[\left(U + gV_1 + gV_2 \right)\phi \right](\underline{x}) \\
&= \sum_{\underline{y}:\, \|\underline{y}-\underline{x}\|=1} \phi(\underline{y}) + \left(U(\underline{x}) + g \sum_{j=1}^{2} V(x_j; \omega) \right) \phi(\underline{x}), \\
&\qquad \underline{x} = (x_1, x_2), \; \underline{y} = (y_1, y_2) \in \mathbb{Z}^d \times \mathbb{Z}^d.
\end{aligned}
\tag{1.1}
$$

Here, $x_j = \left(x_j^{(1)}, \ldots, x_j^{(d)} \right)$ and $y_j = \left(y_j^{(1)}, \ldots, y_j^{(d)} \right)$ stand for coordinate vectors of the j-th particle in $\mathbb{Z}^d$, $j = 1, 2$, and $\| \cdot \|$ is the sup-norm in $\mathbb{R}^d \times \mathbb{R}^d$:

$$
\|x\| = \max_{j=1,2} \, \max_{i=1,\ldots,d} \left| x_j^{(i)} \right|, \quad \underline{x} = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d.
$$

Throughout this paper, the random external potential $V(x; \omega)$, $x \in \mathbb{Z}^d$, is assumed to be real IID, with the common distribution function F on $\mathbb{R}$ satisfying the following condition:

(I) $\forall \; \epsilon > 0$,

$$
s(\epsilon)\big(= s(F, \epsilon) \big) := \sup_{a \in \mathbb{R}} \left(F_V(a + \epsilon) - F_V(a) \right) < \infty.
\tag{1.2I}
$$

Finally, the interaction potential U satisfies the following property:

(II) U is a bounded real function $[0, \infty) \to \mathbb{R}$ obeying

$$
U(\underline{x}) = 0, \quad \text{if } \|x_1 - x_2\| > d.
\tag{1.2II}
$$

The purpose of this paper is to establish the so-called Wegner-type estimates for H. More precisely, these estimates are produced for the eigenvalues of a finite-volume approximation $H_\Lambda \left(= H^{(2)}_{\Lambda,U,V,g}(\omega) \right)$ (i.e., a $|\Lambda| \times \Lambda|$ Hermitian matrix) acting on vectors in $\mathbb{C}^\Lambda$:

$$
\begin{aligned}
H_\Lambda \phi(\underline{x}) &= H^0_\Lambda \phi(\underline{x}) + \left[\left(U + gV_1 + gV_2 \right)_\Lambda \phi \right](\underline{x}) \\
&= \sum_{\underline{y} \in \Lambda: \|\underline{y} - \underline{x}\| = 1} \phi(\underline{y}) + \left(U(\underline{x}) + g \sum_{j=1}^2 V(x_j; \omega) \right) \phi(\underline{x}), \qquad (1.3) \\
&\quad \underline{x} = (x_1, x_2), \ \underline{y} = (y_1, y_2) \in \Lambda \times \Lambda.
\end{aligned}
$$

Here $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}^d$ is a finite set of cardinality $|\Lambda|$. For definiteness, we will focus on the case where Λ is specified as a $\mathbb{Z}^d \times \mathbb{Z}^d$ lattice cube written as the Cartesian product of two $\mathbb{Z}^d$ lattice cubes centred at points $u_1 = \left(u_1^{1)}, \ldots, u_1^{(d)} \right) \in \mathbb{Z}^d$ and $u_2 = \left(u_2^{1)}, \ldots, u_2^{(d)} \right) \in \mathbb{Z}^d$:

$$
\left[\left(\underset{i=1}{\overset{d}{\times}} \left[-L + u_1^{(i)}, u_1^{(i)} + L \right] \right) \times \left(\underset{i=1}{\overset{d}{\times}} \left[-L + u_2^{(i)}, u_2^{(i)} + L \right] \right) \right] \cap \left(\mathbb{Z}^d \times \mathbb{Z}^d \right). \tag{1.4}
$$

A set Λ of the form (1.4) will be called a box and denoted by $\Lambda_L(\underline{u})$, $\underline{u} = (u_1, u_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$, while the $\mathbb{Z}^d$ lattice cubes figuring in the RHS (1.4) as the Cartesian factors will be denoted by $\Pi_1 \Lambda_L(\underline{u})$ and $\Pi_2 \Lambda_L(\underline{u})$:

$$
\Pi_j \Lambda_L(\underline{u}) = \left(\underset{i=1}{\overset{d}{\times}} \left[-L + u_j^{(i)}, u_j^{(i)} + L \right] \right) \cap \mathbb{Z}^d, \ \ j = 1, 2. \tag{1.5}
$$

We will also call cubes $\Pi_1 \Lambda_L(\underline{u})$ and $\Pi_2 \Lambda_L(\underline{u})$ the projections of $\Lambda_L(\underline{u})$. The cardinality of box $\Lambda_L(\underline{u})$ is denoted by $|\Lambda_L(\underline{u})|$ and the cardinality of cube $\Pi_j \Lambda_L(\underline{u})$ by $|\Pi_j \Lambda_L(\underline{u})|$. Symbol $\mathbb{P}$ will stand for the probability distribution generated by random variables $V(x; \omega)$, $x \in \mathbb{Z}^d$. Symbol $\mathfrak{B} \left[\Lambda_L(\underline{u}) \right]$ is used for the sigma-algebra generated by random variables

$$
\omega \mapsto V(x_1; \omega) + V(x_2; \omega), \ \ \underline{x} = (x_1, x_2) \in \Lambda_L(\underline{u}).
$$

The spectrum $\Sigma \left(H_{\Lambda_L(\underline{u})} \right)$ of matrix $H_{\Lambda_L(uu)}$ is a random subset of $\mathbb{R}$ consisting of $|\Lambda_L(\underline{u})|$ points $\lambda^{(k)}_{\Lambda_L(\underline{u})} (= \lambda^{(k)}_{\Lambda_L(\underline{u})}(\omega))$, $k = 1, \ldots, |\Lambda_L(\underline{u})|$ (random eigen-values in volume $\Lambda_L(\underline{u})$, measurable with respect to $\mathfrak{B} \left[\Lambda_L(\underline{u}) \right]$). Given a value $E \in \mathbb{R}$, we denote

$$
\mathrm{dist} \left[\Sigma \left(H_{\Lambda_L(\underline{u})} \right), E \right] = \min \left[\left| E - \lambda^{(k)}_{\Lambda_L(\underline{u})} \right| : \ k = 1, \ldots, |\Lambda_L(\underline{u})| \right]. \tag{1.6}
$$

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Our first result in this paper is the so-called single-volume Wegner bound given in Theorem 1.

Theorem 1. $\forall\ E \in \mathbb{R},\ L > 1,\ \underline{u} \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $\epsilon > 0$,

$$\mathbb{P}\Big(\operatorname{dist} \big[\Sigma \left(H_{\Lambda_L(\underline{u})} \right), E \big] \leq \epsilon \Big) \leq |\Lambda_L(\underline{u})| \ \Big|\Pi_1\Lambda_L(\underline{u}) \cup \Pi_2\Lambda_L(\underline{u})\Big| \cdot s(2\epsilon). \quad (1.7)$$

In Theorem 2 below we deal with a two-volume Wegner bound. This bound assesses the probability that the random spectra $\Sigma\left(H_{\Lambda_L(\underline{u})} \right)$ and $\Sigma\left(H_{\Lambda_L(\underline{u}')} \right)$ are close to each other, for a pair of boxes $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ positioned away from each other, and conditional on sigma-algebra $\mathfrak{B}\left[\Lambda_L(\underline{u}') \right]$. More precisely, set:

$$\operatorname{dist} \big[\Sigma \left(H_{\Lambda_L(\underline{u})} \right), \Sigma \left(H_{\Lambda_L(\underline{u}')} \right) \big] = \min \ \Big[\Big| \lambda^{(k)}_{\Lambda_L(\underline{u}')} - \lambda^{(k')}_{\Lambda_L(\underline{u}')} \Big| :$$
$$k, k' = 1, \ldots, |\Lambda_L(\underline{u})| \Big]. \quad (1.8)$$

Our next result provides a probabilistic estimate on the distance between spectra in two disjoint boxes. An important feature of two-particle operators is that the potential $W(u_1, u_2) = U(u_1, u_2) + g(V(u_1; \omega) + V(u_1; \omega))$ is a symmetric function of the pair $(u_1, u_2) \in \mathbb{Z}^d$. Namely, let $S : \mathbb{Z}^2 \times \mathbb{Z}^d$ be the following symmetry:

$$S : (u_1, u_2) \mapsto (u_2, u_1).$$

Then $W(S(\underline{x})) \equiv W(\underline{x})$. As a consequence, spectra of operators H_Λ and $H_{S(\Lambda)}$ are identical.

Theorem 2. $\forall\ L > 1,\ \underline{u}, \underline{u}' \in \mathbb{Z}^d \times \mathbb{Z}^d$ with

$$\min \left\{ \|\underline{u} - \underline{u}'\|, \|S(\underline{u}) - \underline{u}'\| \right\} \geq 8L \quad (1.9)$$

and $\epsilon > 0$, at least one of the following inequalities holds: either

$$\mathbb{P}\Big(\operatorname{dist} \big[\Sigma \left(H_{\Lambda_L(\underline{u})} \right), \Sigma \left(H_{\Lambda_L(\underline{u}')} \right) \big] \leq \epsilon \Big| \mathfrak{B}\left[\Lambda_L(\underline{u}') \right] \Big)$$
$$\leq |\Lambda_L(\underline{u})| \ |\Lambda_L(\underline{u}')| \ \Big|\Pi_1\Lambda_L(\underline{u}) \cup \Pi_2\Lambda_L(\underline{u})\Big| \cdot s(2\epsilon). \quad (1.10A)$$

or

$$\mathbb{P}\Big(\operatorname{dist} \big[\Sigma \left(H_{\Lambda_L(\underline{u})} \right), \Sigma \left(H_{\Lambda_L(\underline{u}')} \right) \big] \leq \epsilon \Big| \mathfrak{B}\left[\Lambda_L(\underline{u}) \right] \Big)$$
$$\leq |\Lambda_L(\underline{u})| \ |\Lambda_L(\underline{u}')| \ \Big|\Pi_1\Lambda_L(\underline{u}') \cup \Pi_2\Lambda_L(\underline{u}')\Big| \cdot s(2\epsilon). \quad (1.10B)$$

The assertions of Theorems 1 and 2 are proved in the next section of the paper, with the help of the so-called Stollmann's lemma. They are useful in the spectral analysis of H and $H_{\Lambda_L(\underline{u})}$. See [4]. Note that in Theorem 1 we deal with the probability distribution $\mathbb{P}_{\Lambda_L(\underline{u})}$ generated by the random variables

$$\omega \mapsto V(x, \omega), \ x \in \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}), \tag{1.11}$$

whereas in Theorem 2 it is the conditional probability distribution $\mathbb{P}_{\Lambda_L(\underline{u}), \Lambda_L(\underline{u}')}\big(\ \cdot \ \big| \mathfrak{B}\left[\Lambda_L(\underline{u}')\right]\big)$ generated by

$$V(x, \ \cdot \ ), \ x \in \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}) \cup \Pi_1 \Lambda_L(\underline{u}') \cup \Pi_2 \Lambda_L(\underline{u}') \tag{1.12}$$

and conditioned relative to $\mathfrak{B}\left[\Lambda_L(\underline{u}')\right]$.

Throughout the paper, symbol $\square$ is used to mark the end of a proof.

2 Stollmann's lemma. Proof of Theorems 1 and 2

2.1 Stollmann's lemma and its use

For reader's convenience, we provide here the statement of Stollmann's lemma and its proof; see Lemma 2.1 below. Cf. [3] and [4], Lemma 2.3.1. Let Π be a non-empty finite set of cardinality $|\Pi| = p$. We assume that Π is ordered and identify it with the set $\{1, 2, \ldots, p\}$. Consider the Euclidean space $\mathbb{R}^\Pi$ of real dimension p, with standard basis $(\mathbf{e}_1, \ldots, \mathbf{e}_p)$, and its positive orthant

$$\mathbb{R}_+^\Pi = \big\{ \mathbf{v} = (q_1, \ldots, q_p) \in \mathbb{R}^\Pi : q_j \geq 0, \ j = 1, \ldots, p \big\}.$$

For a given probability measure μ on $\mathbb{R}$, denote by μ^Π the product measure $\mu \times \cdots \times \mu$ on $\mathbb{R}^\Pi$ and by $\mu^{\Pi \setminus \{1\}}$ be the marginal product measure induced by μ^Π on $\mathbb{R}^{\Pi \setminus \{1\}}$. Next, $\forall \ \epsilon > 0$ set

$$s(\mu, \epsilon) = \sup_{a \in \mathbb{R}} \int_a^{a+\epsilon} d\mu(t) \tag{2.1}$$

and assume that $s(\mu, \epsilon) < \infty$.

Definition 2.1. *A function* $\Phi : \mathbb{R}^\Pi \to \mathbb{R}$ *is called diagonally-monotone (DM) if it satisfies the following conditions:*

(i) $\forall \; \mathbf{r} \in \mathbb{R}_+^p$ *and any* $\mathbf{v} \in \mathbb{R}^p$,

$$\Phi(\mathbf{v} + \mathbf{r}) \geq \Phi(\mathbf{v}); \tag{2.2}$$

(ii) *moreover, with vector* $\mathbf{e} = \mathbf{e}_1 + \ldots + \mathbf{e}_p \in \mathbb{R}^p$, $\forall \; \mathbf{v} \in \mathbb{R}^p$ *and* $t > 0$

$$\Phi(\mathbf{v} + t\mathbf{e}) - \Phi(\mathbf{v}) \geq t. \tag{2.3}$$

Lemma 2.1. *Suppose function* $\Phi : \mathbb{R}^\Pi \to \mathbb{R}$ *is DM. Then* $\forall \; \epsilon > 0$ *and any open interval* $I \subset \mathbb{R}$ *of length* ϵ,

$$\mu^\Pi \{ \mathbf{v} : \; \Phi(\mathbf{v}) \in I \} \leq p \cdot s(\mu, \epsilon). \tag{2.4}$$

Proof. Let $I = (a, b)$, $b - a = \epsilon$, and consider the set

$$A = \{ \mathbf{v} : \; \Phi(\mathbf{v}) \leq a \}.$$

Furthermore, define recursively sets A_j^ϵ, $j = 0, \ldots, p$, by setting

$$A_0^\epsilon = A, \; A_j^\epsilon = A_{j-1}^\epsilon + [0, \epsilon] \mathbf{e}_j := \{ \mathbf{v} + t\mathbf{e}_j : \; \mathbf{v} \in A_{j-1}^\epsilon, \; t \in [0, \epsilon] \}.$$

Obviously, the sequence of sets A_j^ϵ, $j = 1, 2, ...$, is increasing with j. The DM property implies that $\{ \mathbf{v} : \; \Phi(\mathbf{v}) < b \} \subset A_p^\epsilon$. Indeed, if $\Phi(\mathbf{v}) < b$, then for the vector $\mathbf{v}' := \mathbf{v} - \epsilon \cdot \mathbf{e}$ we have by property (ii):

$$\Phi(\mathbf{v}') \leq \Phi(\mathbf{v}' + \epsilon \cdot \mathbf{e}) - \epsilon = \Phi(\mathbf{v}) - \epsilon \leq b - \epsilon \leq a,$$

meaning that $\mathbf{v}' \in \{ \Phi \leq a \} = A$ and, therefore, $\mathbf{v} = \mathbf{v}' + \epsilon \cdot \mathbf{e} \in A_p^\epsilon$. We conclude that

$$\{ \mathbf{v} : \; \Phi(\mathbf{v}) \in I \} = \{ \mathbf{v} : \; \Phi(\mathbf{v}) < b \} \setminus \{ \mathbf{v} : \; \Phi(\mathbf{v}) \leq a \} \subset A_m^\epsilon \setminus A.$$

Moreover, the probability $\mu^\Pi \{ \mathbf{v} : \; \Phi(\mathbf{v}) \in I \}$ is

$$\leq \mu^\Pi \left(A_p^\epsilon \setminus A \right) = \mu^\Pi \left(\bigcup_{j=1}^p \left(A_j^\epsilon \setminus A_{j-1}^\epsilon \right) \right) \leq \sum_{j=1}^p \mu^\Pi \left(A_j^\epsilon \setminus A_{j-1}^\epsilon \right).$$

For $\widetilde{\mathbf{v}} \in \mathbb{R}^{\Pi \setminus \{1\}}$, set $I_1(\widetilde{\mathbf{v}}) = \{ q_1 \in \mathbb{R} : \; (q_1, \widetilde{\mathbf{v}}) \in A_1^\epsilon \setminus A \}$. Then, by definition of set A_1^ϵ, set $I_1(\widetilde{\mathbf{v}})$ is an interval of length $\leq \epsilon$. Thus,

$$\mu^\Pi (A_1^\epsilon \setminus A) = \int d\mu^{\Pi \setminus \{1\}}(\widetilde{\mathbf{v}}) \int_{I_1(\widetilde{\mathbf{v}})} d\mu(q_1) \leq s(\mu, \epsilon).$$

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Similarly, for $j = 2, \ldots, p$ we obtain $\mu^{\Pi}(A_j^{\epsilon} \setminus A_{j-1}^{\epsilon}) \leq s(\mu, \epsilon)$, which yields that

$$\mu^{\Pi}\{\mathbf{v} : \Phi(\mathbf{v}) \in I\} \leq \sum_{j=1}^{p} \mu^{\Pi}(A_j^{\epsilon} \setminus A_{j-1}^{\epsilon}) \leq p \cdot s(\mu, \epsilon). \qquad \square$$

In our situation, it is also convenient to introduce the notion of a DM operator family.

Definition 2.2. *Let $\mathcal{H}$ be a Hilbert space of a finite dimension m. A family of Hermitian operators $B(\mathbf{v}) : \mathcal{H} \to \mathcal{H}$, $\mathbf{v} \in \mathbb{R}^{\Pi}$, is called DM if, $\forall\, f \in \mathcal{H}$*

$$(B(\mathbf{v} + t \cdot \mathbf{e})f, f) - (B(\mathbf{v})f, f) \geq t \cdot \|f\|^2. \tag{2.5}$$

That is, $\forall\, f \in \mathcal{H}$ with $\|f\| = 1$, the function $\Phi_f : \mathbb{R}^{\Pi} \to \mathbb{R}$ defined by $\Phi_f(\mathbf{v}) = (B(\mathbf{v})f, f)$ is DM.

Remark 2.1. Suppose that $B(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^{\Pi}$, is a DM operator family in $\mathcal{H}$. Let $E_{B(\mathbf{v})}^{(1)} \leq \ldots \leq E_{B(\mathbf{v})}^{(m)}$ be the eigen-values of $B(\mathbf{v})$. Then, by virtue of the variational principle, $\forall\, k = 1, \ldots, m$, $\mathbf{v} \mapsto E_{B(\mathbf{v})}^{(k)}$ is a DM function.

Remark 2.2. If $B(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^{\Pi}$, is a DM operator family in $\mathcal{H}$, and $K : \mathcal{H} \to \mathcal{H}$ is an arbitrary Hermitian operator, then the family $K + B(\mathbf{v})$ is also DM.

2.2 Proof of Theorems 1 and 2

Proof of Theorem 1. The proof is a straightforward application of Lemma 2.1 and Remarks 2.1 and 2.2. Cf. the proof of Theorems 2.3.2 and 2.3.3 in [2]. For a single-particle tight binding model, similar results are presented in [1]. In our situation, set Π is identified as the union $\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})$, with $p = |\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})|$. Vector $\mathbf{v}$ is identified with a collection $\{V(x, \omega),\ x \in \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})\}$ of sample values of the external potential; to stress this fact we will write

$$\mathbf{v} \sim \{V(x, \omega),\ x \in \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})\}. \tag{2.6}$$

Next, probability measure μ represents the distribution of a single value, say $V(0, \cdot)$, and product-measure μ^{Π} is identified as $\mathbb{P}_{\Lambda_L(\underline{u})}$. Further, the Hilbert space $\mathcal{H}$ in Remarks 2.1 and 2.2 is $\mathbb{C}^{\Lambda_L(\underline{u})}$, of dimension $m = |\Lambda_L(\underline{u})|$, in which

the action of matrix $H_{\Lambda_L(\underline{u})}$ is considered. Given $\underline{x} = (x_1, x_2) \in \Lambda_L(\underline{u})$, we can write

$$g\big[V(x_1, \omega) + V(x_2, \omega)\big] = \sum_{y \in \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})} c(\underline{x}, y) V(y, \omega)$$

where $c(\underline{x}, y) = \begin{cases} 1, & y = x_1 \text{ or } x_2, \\ 0, & y \neq x_1, x_2. \end{cases}$ This implies that, with identification (2.6), operators Hermitian $B(\mathbf{v})$ form a DM family. Here $B(\mathbf{v})$ is the multiplication operator

$$B(\mathbf{v})\phi(\underline{x}) = g\big[V(x_1, \omega) + V(x_2, \omega)\big]\phi(x), \quad \underline{x} \in \Lambda_L(\underline{u}), \ \phi \in \mathbb{C}^{\Lambda_L(\underline{u})} \qquad (2.7)$$

Then we use Remark 2.2, with $K = H^0_{\Lambda_L(\underline{u})} + U$ (cf. (1.3)), and obtain that $H_{\Lambda_L(\underline{u})} = K + B(\mathbf{v})$ is a DM family. Next, owing to Remark 2.1, each eigen-value $\lambda^{(k)}_{\Lambda_L(\underline{u})}$, $k = 1, \ldots, |\Lambda_L(\underline{u})|$, is a DM function of the sample collection $\{V(x, \omega), \ x \in \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})\}$. Hence, by Lemma 2.1, $\forall$ $k = 1, \ldots, |\Lambda_L(\underline{u})|\,\big]$,

$$\mathbb{P}\Big(\Big| E - \lambda^{(k)}_{\Lambda_L(\underline{u})} \Big| \leq \epsilon \Big) \leq |\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})|\, s(F, 2\epsilon), \qquad (2.8)$$

The final remark is, that the probability in the LHS of Eqn (1.7) is $\leq$ the RHS of Eqn (2.8) times $|\Lambda_L(\underline{u})|$. $\qquad \square$

We will need the following elementary geometrical statement.

Lemma 2.2. *Consider two boxes $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ and suppose that*

$$\min(\|\underline{u} - \underline{u}'\|, \|S(\underline{u}) - \underline{u}'\|) \geq 8L \qquad (2.9)$$

Then there are two possibilities (which in general do not exclude each other):
  (i) $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ are 'completely separated', when

$$(\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})) \cap (\Pi_1 \Lambda_L(\underline{u}') \cup \Pi_2 \Lambda_L(\underline{u}')) = \emptyset. \qquad (2.10)$$

(ii) $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ are 'partially separated'. In this case one (or more) of the four possibilities can occur:

$$
\begin{array}{lll}
\text{(A)} & \Pi_1 \Lambda_L(\underline{u}) \cap \big[\Pi_2 \Lambda_L(\underline{u}) \cup \Pi \Lambda_L(\underline{u}')\big] & = \emptyset, \\
\text{(B)} & \Pi_2 \Lambda_L(\underline{u}) \cap \big[\Pi_1 \Lambda_L(\underline{u}) \cup \Pi \Lambda_L(\underline{u}')\big] & = \emptyset, \\
\text{(C)} & \Pi_1 \Lambda_L(\underline{u}') \cap \big[\Pi \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}')\big] & = \emptyset, \\
\text{(D)} & \Pi_2 \Lambda_L(\underline{u}') \cap \big[\Pi \Lambda_L(\underline{u}) \cup \Pi_1 \Lambda_L(\underline{u}')\big] & = \emptyset,
\end{array}
\qquad (2.11)
$$

where

$$\Pi\Lambda_L(\underline{u}') = \Pi_1\Lambda_L(\underline{u}') \cup \Pi_2\Lambda_L(\underline{u}'), \quad \Pi\Lambda_L(\underline{u}) = \Pi_1\Lambda_L(\underline{u}) \cup \Pi_2\Lambda_L(\underline{u}). \quad (2.12)$$

Pictorially, case (ii) is where one of the cubes $\Pi_j\Lambda_L(\underline{u})$, $\Pi_j\Lambda_L(\underline{u}')$, $j = 1, 2$, is disjoint from the union of the rest of the projections of $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$.

Proof of Theorem 2. Owing to Lemma 2.2, boxes $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ satisfy either (i) or (ii), i.e. they are either completely or partially separated. We note that the use of the max-norm $\| \ \|$ is convenient here as it leads to the constant 8 (equal to 2 times 4, the number of projections $\Pi_j\Lambda_L(\underline{u})$ and $\Pi_j\Lambda_L(\underline{u}')$, $j = 1, 2$) which does not depend on the dimension d.

Passing to the proof of Theorem 2 proper, note first that, under the conditional probability distribution in Eqn (1.10A), the eigen-values $\lambda_{\Lambda_L(\underline{u}')}^{(k')}$, $k' = 1, \ldots, |\Lambda_L(\underline{u}')|$, forming the set $\Sigma\left(H_{\Lambda_L(\underline{u}')}\right)$ are non-random. The same is true, of course, for the eigen-values $\lambda_{\Lambda_L(\underline{u}')}^{(k')}$, $k' = 1, \ldots, |\Lambda_L(\underline{u}')|$ in Eqn (1.10B). Now, by virtue of (2.10), (2.11), the boxes $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ are either completely or partially separated. In the former case, the conditional probability distribution in (1.9) is reduced to the probability measure $\mathbb{P}_{\Lambda_L(\underline{u})}$. Then, as in the proof of Theorem 1 (cf. Eqn (2.8)), $\forall \ k = 1, \ldots, |\Lambda_L(\underline{u}')|$, for the (random) eigen-value $\lambda_{\Lambda_L(\underline{u})}^{(k)} \in \Sigma\left(H_{\Lambda_L(\underline{u})}\right)$ the following bound holds true:

$$\mathbb{P}\left(\left|\lambda_{\Lambda_L(\underline{u}')}^{(k)} - \lambda_{\Lambda_L(\underline{u}')}^{(k')}\right| \le \epsilon \Big| \mathfrak{B}\left[\Lambda_L(\underline{u}')\right] \right) \le |\Pi_1\Lambda_L(\underline{u}) \cup \Pi_2\Lambda_L(\underline{u})| \, s(F, 2\epsilon), \tag{2.13}$$

implying bound (1.10A).

Now assume that $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ are partially separated. For example, assume case A where $\Pi_1\Lambda_L(\underline{u})$, is disjoint from the union of the rest of the projections of $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$:

$$\Pi_1\Lambda_L(\underline{u}) \cap [\Pi_2\Lambda_L(\underline{u}) \cup \Pi\Lambda_L(\underline{u}'))] = \emptyset. \tag{2.14}$$

We then write the probability in the LHS of (2.13) as the conditional expectation

$$\mathbb{P}\left(\left|\lambda_{\Lambda_L(\underline{u}')}^{(k)} - \lambda_{\Lambda_L(\underline{u}')}^{(k')}\right| \le \epsilon \Big| \mathfrak{B}\left[\Lambda_L(\underline{u}')\right] \right)$$
$$= \mathbb{E}\left[\mathbb{P}\left(\left|\lambda_{\Lambda_L(\underline{u}')}^{(k)} - \lambda_{\Lambda_L(\underline{u}')}^{(k')}\right| \le \epsilon \Big| \mathfrak{C}\left[\Pi_2\Lambda_L(\underline{u}) \cup \Pi\Lambda_L(\underline{u}')\right] \right) \Big| \mathfrak{B}\left[\Lambda_L(\underline{u}')\right] \right]. \tag{2.15}$$

Here $\mathfrak{C}\big[\Pi_2\Lambda_L(\underline{u}) \cup \Pi\Lambda_L(\underline{u}')\big]$ is the sigma-algebra generated by the random variables

$$\omega \mapsto V(x,\omega), \quad x \in \Pi_2\Lambda_L(\underline{u}) \cup \Pi\Lambda_L(\underline{u}');$$

owing to (2.14) it is independent of the sigma-algebra $\mathfrak{C}\,[\Pi_1\Lambda_L(\underline{u})]$ generated by the random variables

$$\omega \mapsto V(x,\omega), \quad x \in \Pi_1\Lambda_L(\underline{u}).$$

We see that the argument used in the proof of Theorem 1 is still applicable, if we replace the product-measure $\mathbb{P}_{\Lambda_L(\underline{u})}$ by its restriction to $\mathfrak{C}\,[\Pi_1\Lambda_L(\underline{u})]$ (which again can be taken as a product-measure μ^Π from Lemma 2.1, with $p = |\Pi_1\Lambda_L(\underline{u})|$). This allows us to write

$$\mathbb{P}\Big(\,\Big|\lambda^{(k)}_{\Lambda_L(\underline{u}')} - \lambda^{(k')}_{\Lambda_L(\underline{u}')}\Big| \leq \epsilon\,\Big|\mathfrak{C}\big[\Pi_2\Lambda_L(\underline{u}) \cup \Pi\Lambda_L(\underline{u}')\big]\Big) \leq |\Pi_1\Lambda_L(\underline{u})|\,s(F,2\epsilon) \tag{2.16}$$

and deduce a similar bound for the the conditional probability in the LHS of (2.15). Inequality (1.10A) is then derived in the standard manner.

If, instead of (2.14), we have one of the other disjointedness relations (B)-(D) in Eqn (2.11) then the argument is conducted in a similar fashion. Naturally, in the case (B) we still prove (1.10A), while in the cases (C) and (D) we prove (1.10B).

This concludes the proof of Theorem 2.

2.3 Proof of Lemma 2.2

Recall that we have two boxes, $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$, satisfying the condition (2.9):

$$\min(\|\underline{u} - \underline{u}'\|, \|S(\underline{u}) - \underline{u}'\|) \geq 8L.$$

Notice that this can be viewed as lower bound for the distance in the factor space $\mathbb{Z}^d \times \mathbb{Z}^d/S$; recall that $S(u_1, u_2) = (u_2, u_1)$.

Since diam $\Lambda_L(\underline{u}) = $ diam $\Lambda_L(\underline{u}') = 2L$, this implies that the union of the four coordinate projections,

$$\Pi_1\Lambda_L(\underline{u}), \Pi_2\Lambda_L(\underline{u}), \Pi_1\Lambda_L(\underline{u}'), \Pi_2\Lambda_L(\underline{u}')$$

cannot be connected. Therefore, it can be decomposed into two or more connected components. Cases (A), (B), (C) and (D) in the statement of Lemma 2.2 correspond to the situation where one of these coordinate projections is

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disjoint with the three remaining projections. So, it suffices to analyse the case where each connected component of the union

$$\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}) \cup \Pi_1 \Lambda_L(\underline{u}') \cup \Pi_2 \Lambda_L(\underline{u}') \tag{2.17}$$

contains exactly two coordinate projections. Furthermore, it suffices to show that the only possible case is (2.10) where

$$(\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u})) \cap (\Pi_1 \Lambda_L(\underline{u}') \cup \Pi_2 \Lambda_L(\underline{u}')) = \emptyset.$$

To do so, we have to exclude two remaining cases, namely,

$$\begin{cases} (\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_1 \Lambda_L(\underline{u}')) \cap (\Pi_2 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}')) = \emptyset \\ \Pi_1 \Lambda_L(\underline{u}) \cap \Pi_1 \Lambda_L(\underline{u}') \neq \emptyset \\ \Pi_2 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}') \neq \emptyset \end{cases} \tag{2.18}$$

and

$$\begin{cases} (\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}')) \cap (\Pi_1 \Lambda_L(\underline{u}') \cup \Pi_2 \Lambda_L(\underline{u})) = \emptyset \\ \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}') \neq \emptyset \\ \Pi_1 \Lambda_L(\underline{u}') \cup \Pi_2 \Lambda_L(\underline{u}) \neq \emptyset \end{cases} \tag{2.19}$$

First, observe that (2.18) contradicts the assumption that $\Lambda_L(\underline{u})$ and $\Lambda_L(\underline{u}')$ are disjoint (and even distant). Indeed, in such a case, there exist lattice points

$$v_1 \in \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}'), \ v_2 \in \Pi_2 \Lambda_L(\underline{u}) \cup \Pi_2 \Lambda_L(\underline{u}'),$$

so that

$$\exists (v_1, v_2) \in [\Pi_1 \Lambda_L(\underline{u}) \times \Pi_2 \Lambda_L(\underline{u})] \cap [\Pi_1 \Lambda_L(\underline{u}') \times \Pi_2 \Lambda_L(\underline{u}')] \\ = \Lambda_L(\underline{u}) \cap \Lambda_L(\underline{u}') = \emptyset,$$

which is impossible.

The case (2.19) can be reduced to (2.18), by the symmetry S. Namely, let $\underline{u}'' = S(\underline{u}')$, then

$$\Pi_1 \Lambda_L(\underline{u}'') = \Pi_2 \Lambda_L(\underline{u}'), \ \Pi_2 \Lambda_L(\underline{u}'') = \Pi_1 \Lambda_L(\underline{u}').$$

Now (2.19) reads as follows in terms of boxes $\Lambda_L(\underline{u})$, $\Lambda_L(\underline{u}')$:

$$\begin{cases} (\Pi_1 \Lambda_L(\underline{u}) \cup \Pi_1 \Lambda_L(\underline{u}'')) \cap (\Pi_2 \Lambda_L(\underline{u}'') \cup \Pi_2 \Lambda_L(\underline{u})) = \emptyset \\ \Pi_1 \Lambda_L(\underline{u}) \cup \Pi_1 \Lambda_L(\underline{u}'') \neq \emptyset \\ \Pi_2 \Lambda_L(\underline{u}'') \cup \Pi_2 \Lambda_L(\underline{u}) \neq \emptyset. \end{cases} \tag{2.20}$$

The same argument as above shows then that $\Lambda_L(\underline{u}) \cap \Lambda_L(\underline{u}'') \neq \emptyset$, which is impossible, since

$$\mathrm{dist}(\underline{u}, S(\underline{u}')) > 8L.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 Concluding remarks

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